Sensor network localization has benign landscape under mild rank relaxation

February 11, 2025

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with
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EPFL

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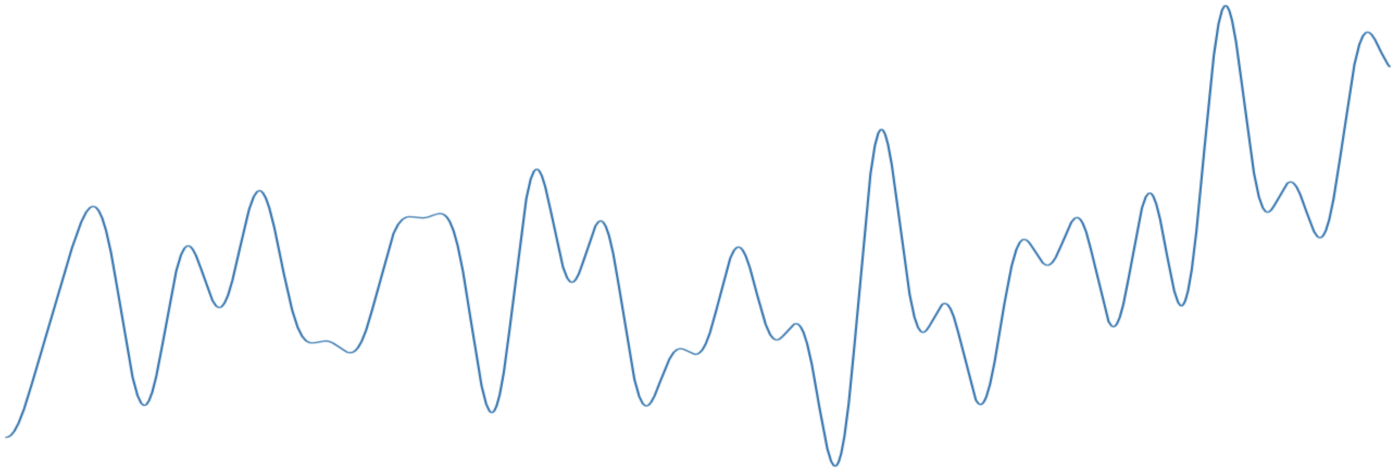

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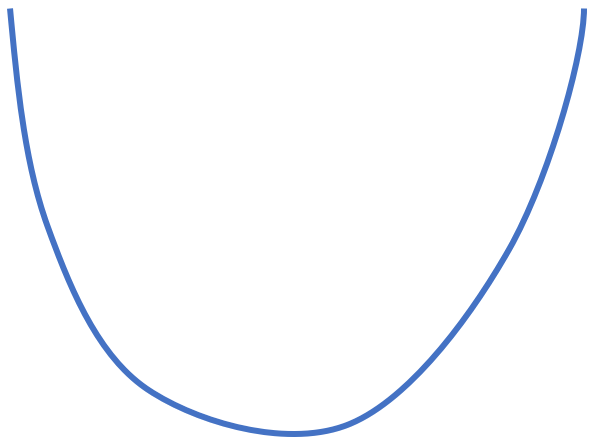

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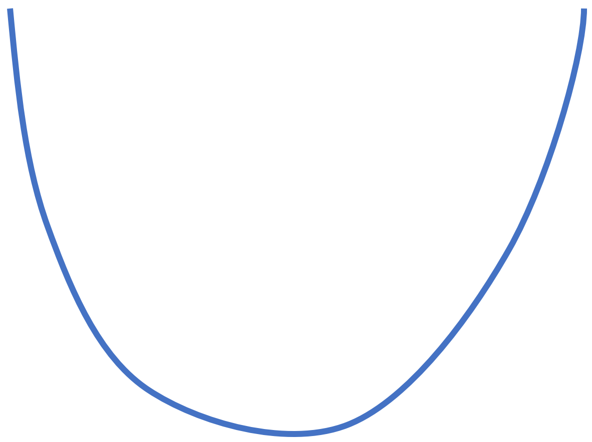

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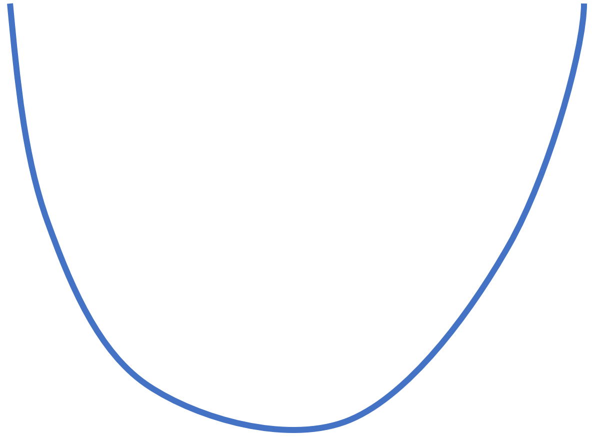

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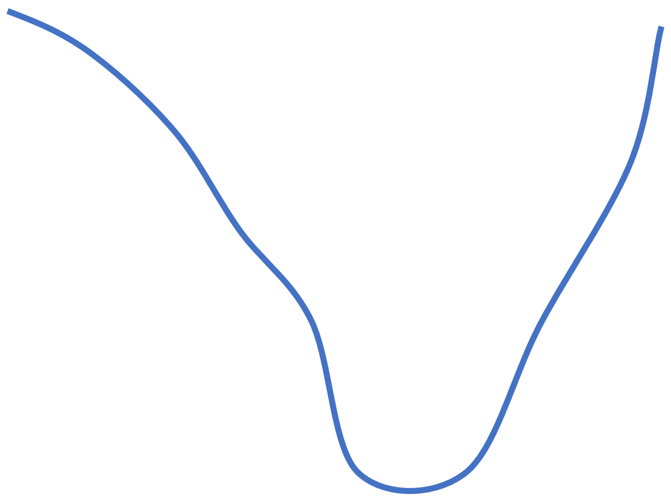

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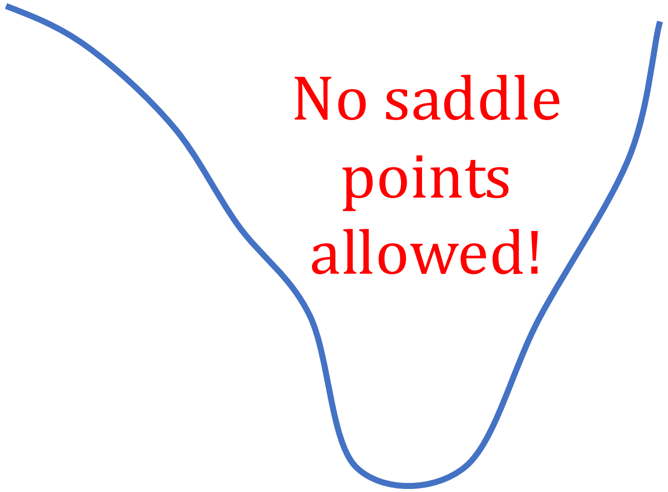

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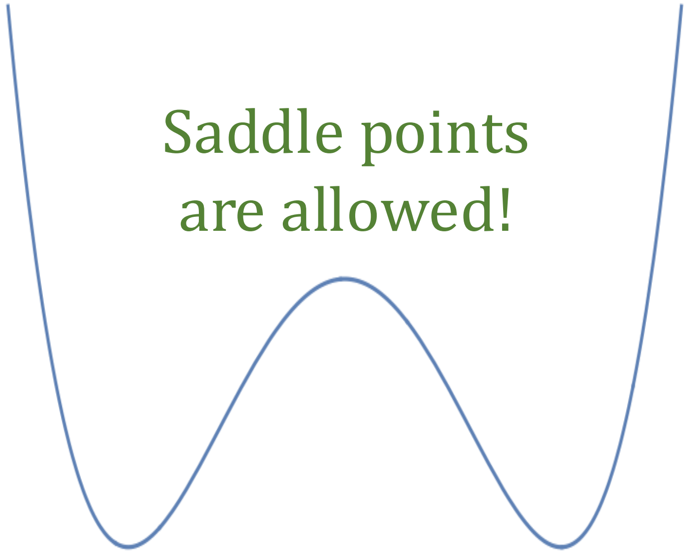
Benign landscape (all local minima are optimal)



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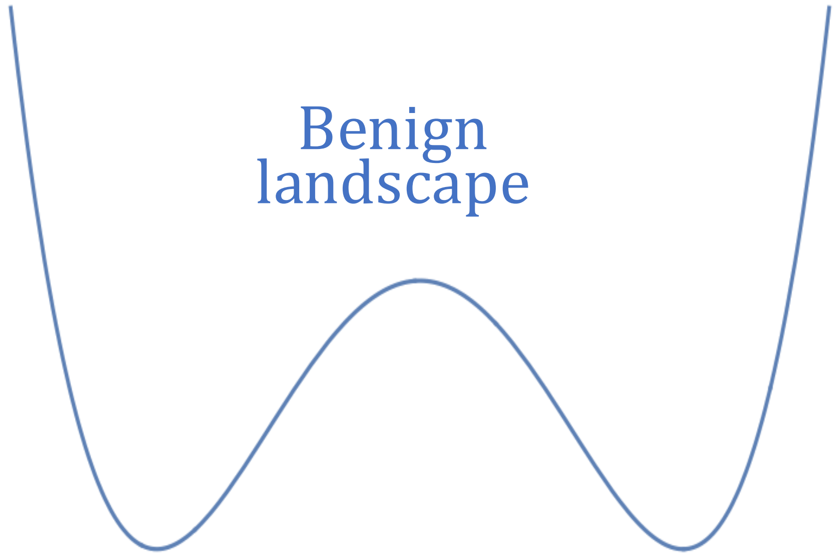

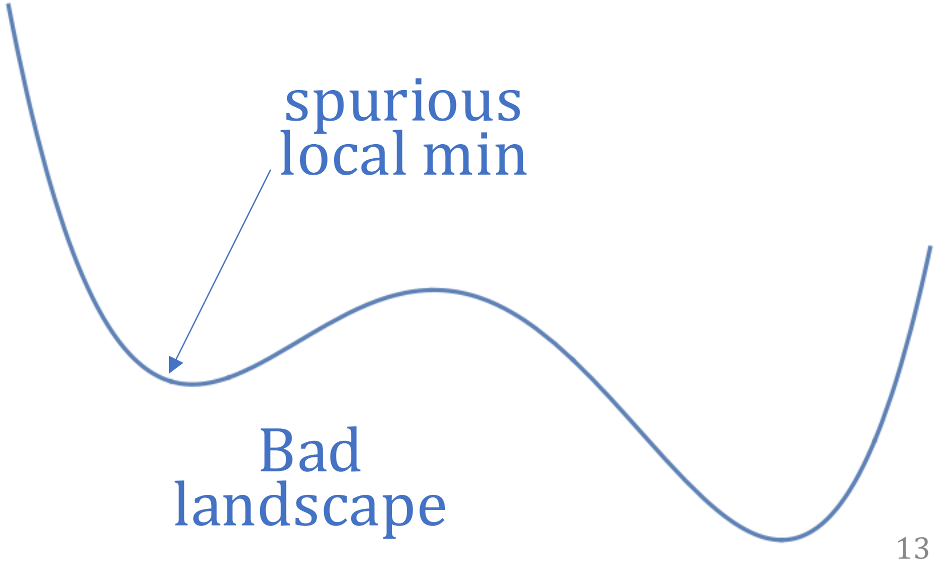

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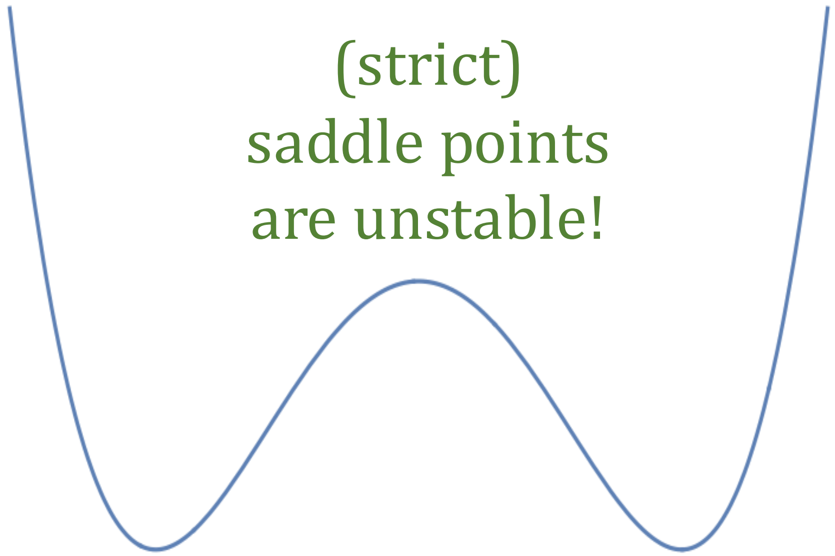

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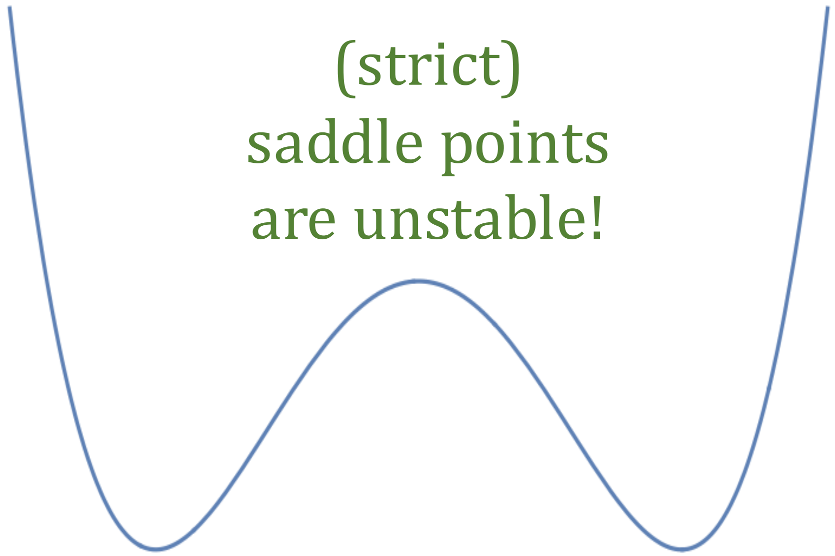
converges to 2-critical point (w/ prob 1)



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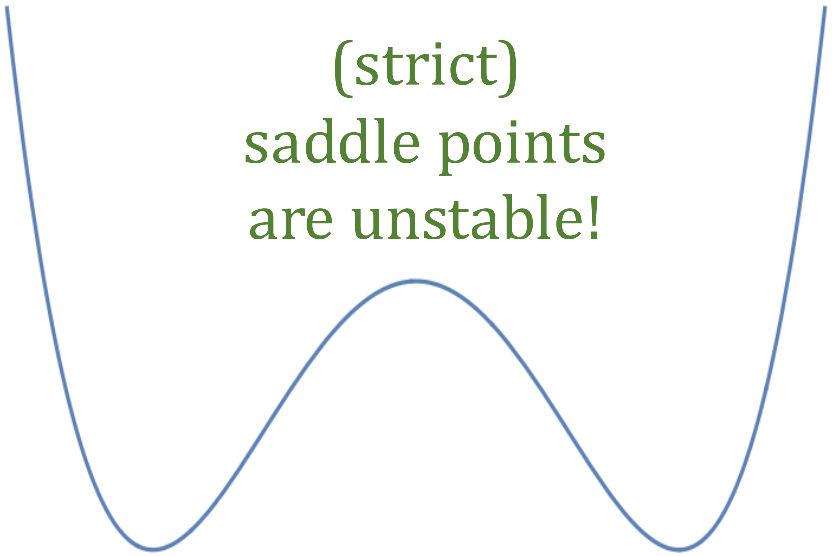
Stable manifold theorems
+
Łojasiewicz theorem



Some of my previous work …

"Negative curvature obstructs acceleration for g-convex optimization"

C, Boumal, 2022

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C, Boumal, 2023

Invexity

"Synchronization on circles and spheres with nonlinear interaction"

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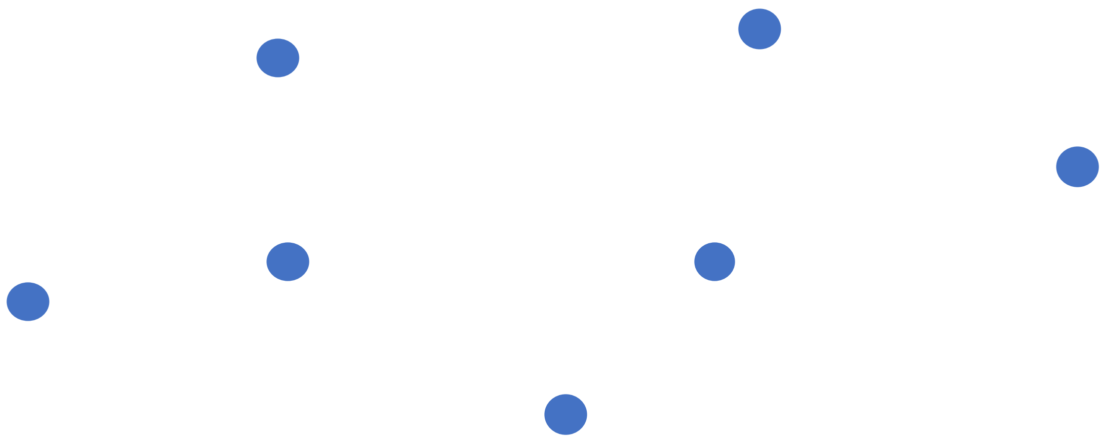
The problem: SNL

n unknown points $z_1^*, z_2^*, \ldots, z_n^*$ in $\mathbb{R}^\ell$.



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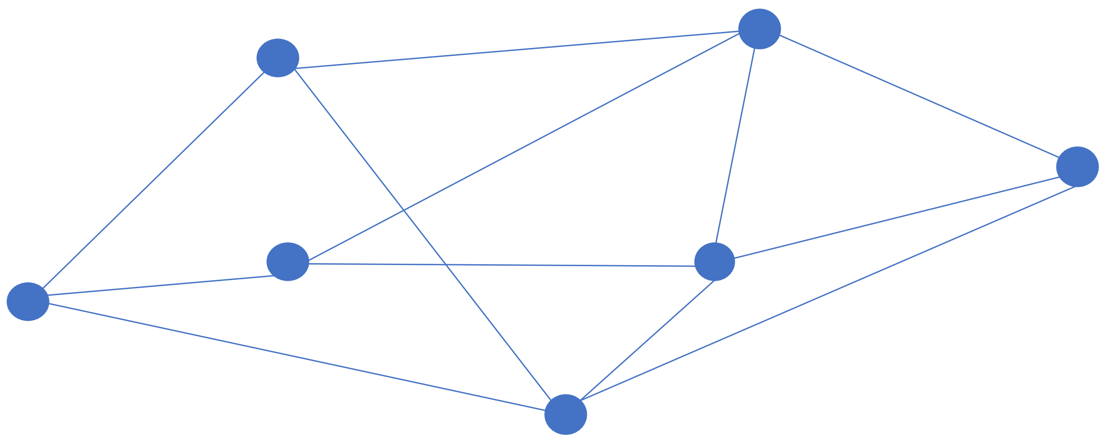

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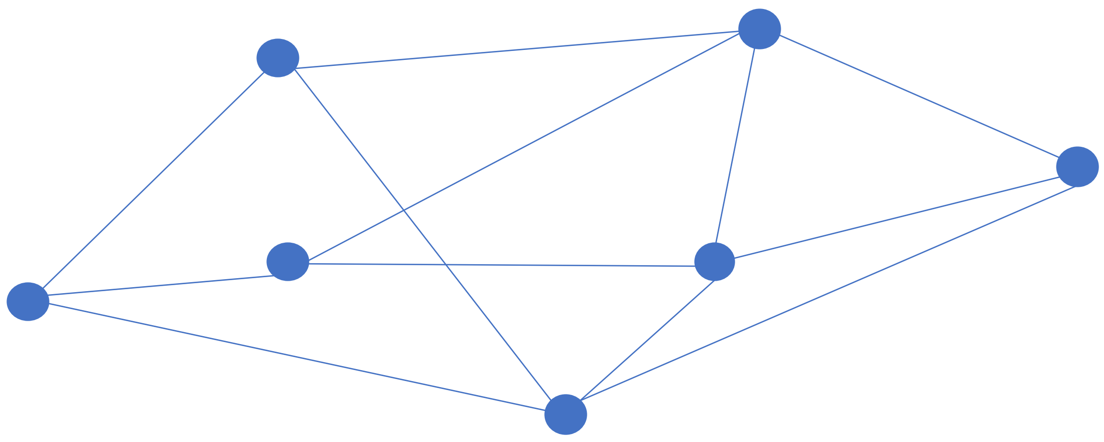

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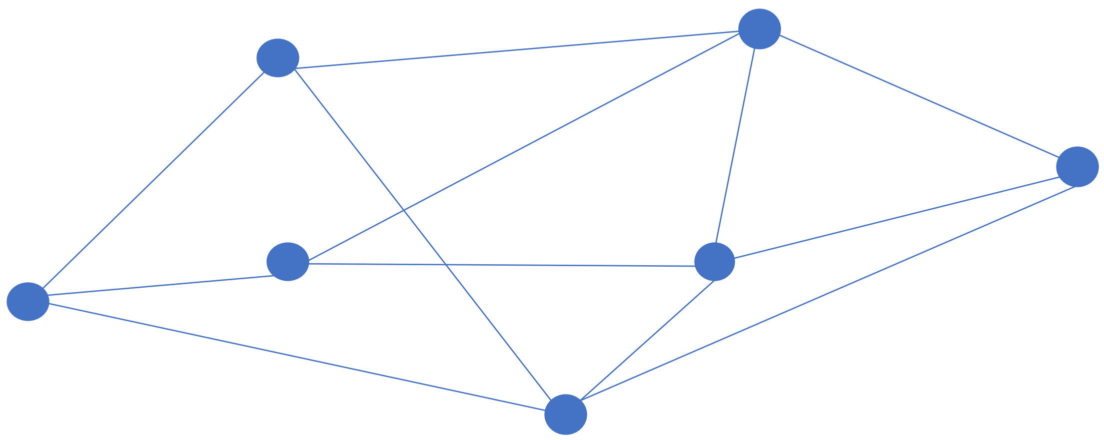
Sensor network localization (SNL) – Torgerson ’58, Shepard ‘62



Applications

Robotics (**sensor network localization**), ℓ = dimension = 2,3

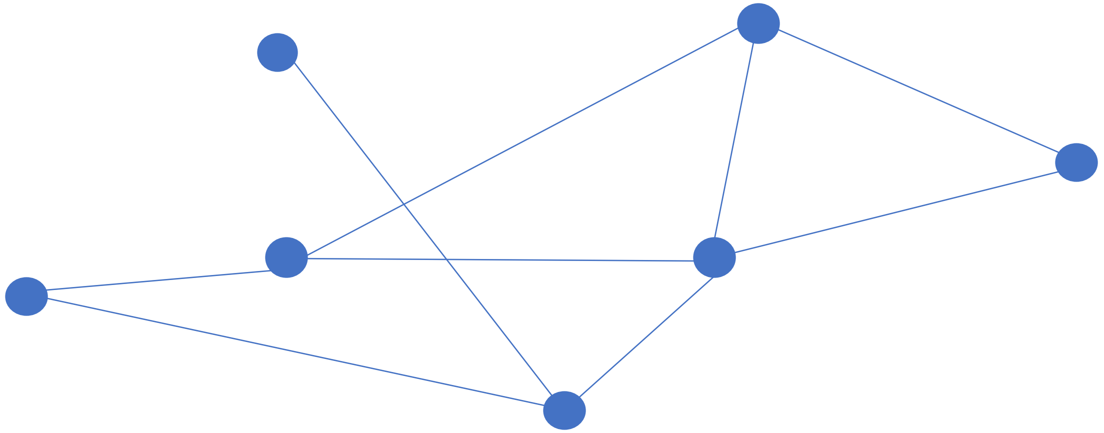
Molecular conformation

Data analysis (metric **multidimensional scaling**)

Graph theory (rigidity)

The problem: SNL

Need enough distance measurements for well-posedness (*global rigidity*)



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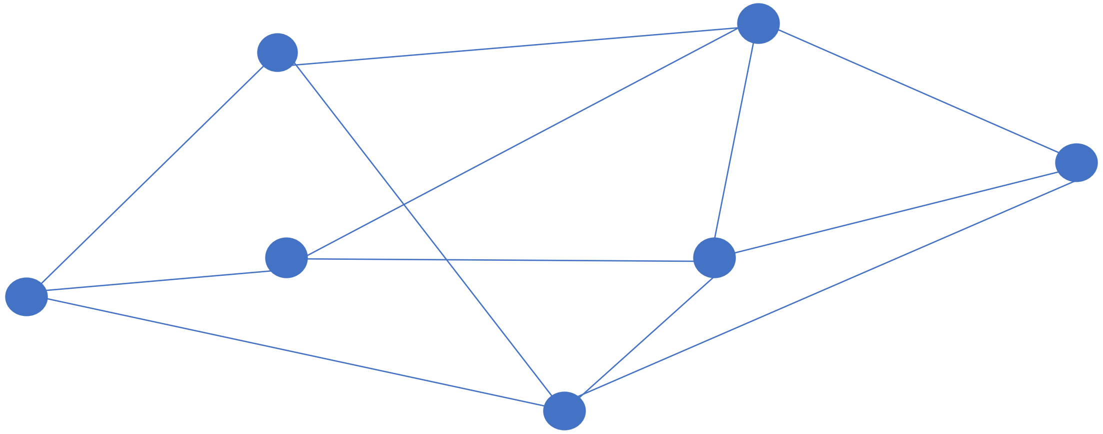

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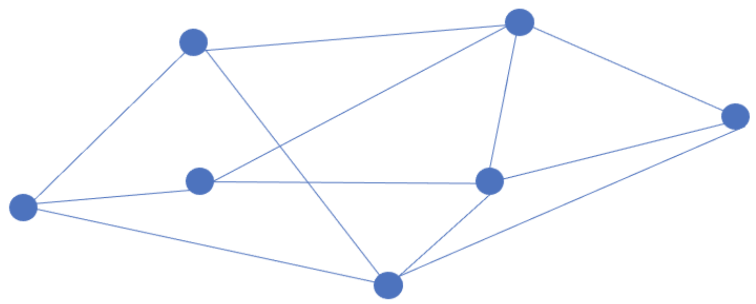

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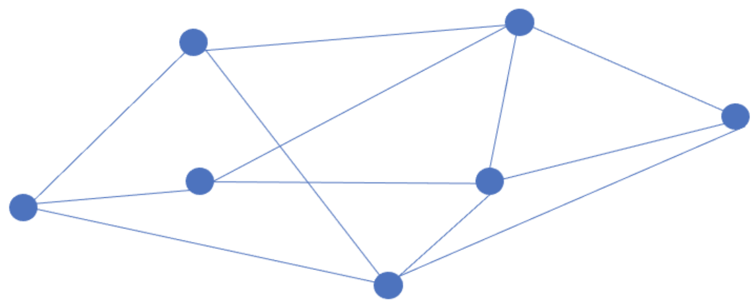

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“Theory of semidefinite programming for Sensor Network Localization” -- So, Ye ‘06

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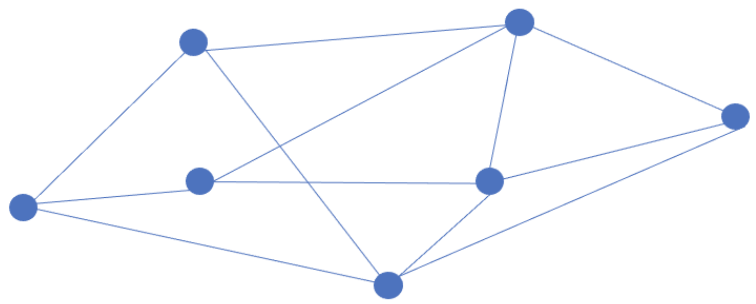

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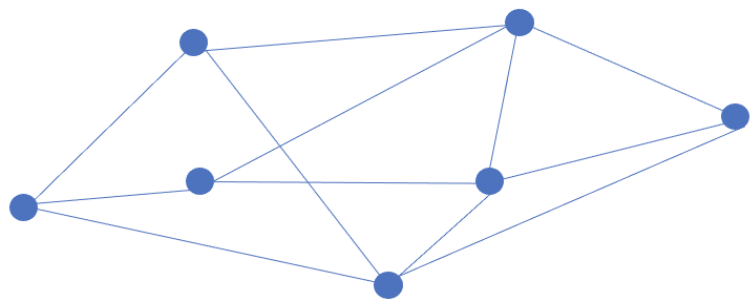

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Variable size = $n\ell$ = **linear in # of points**

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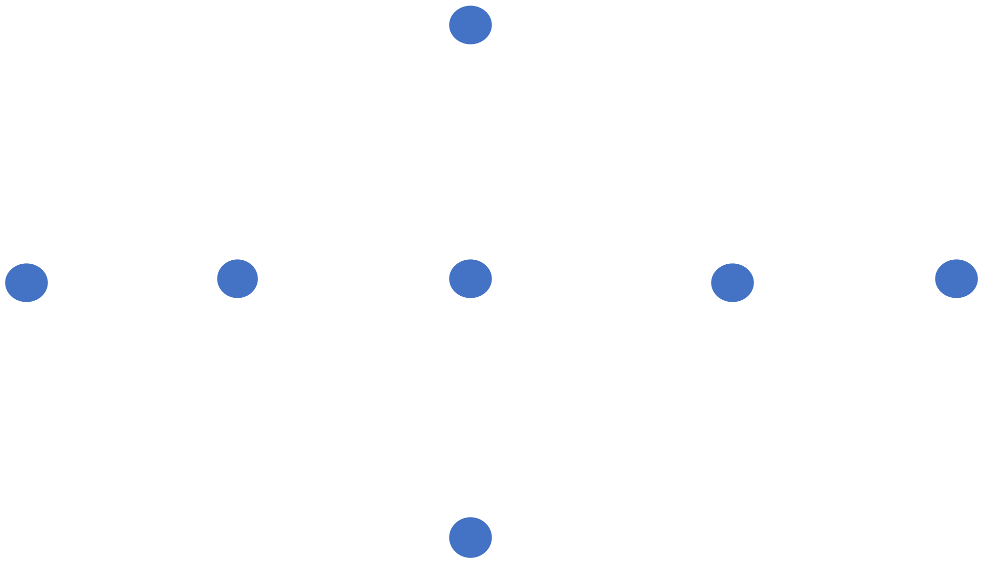
s-stress can have spurious strict local minima!

Ground truth $z_1^*, z_2^*, \ldots$



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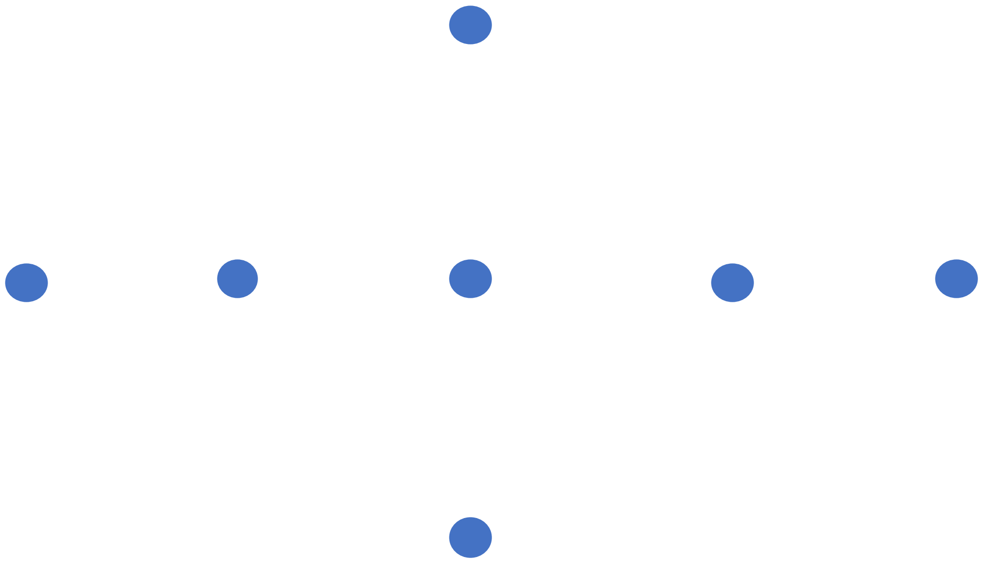

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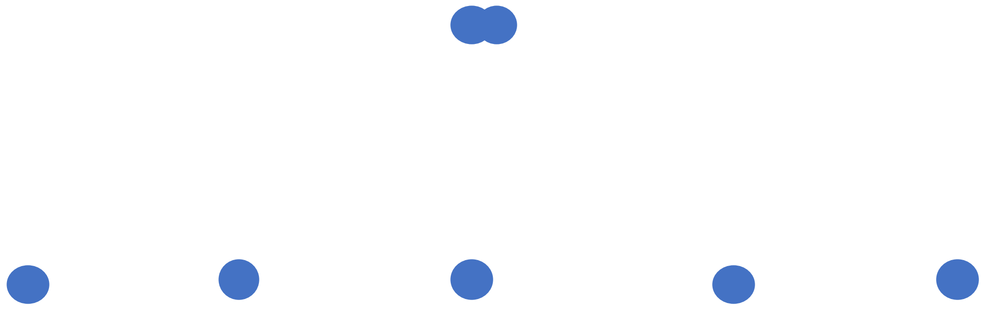

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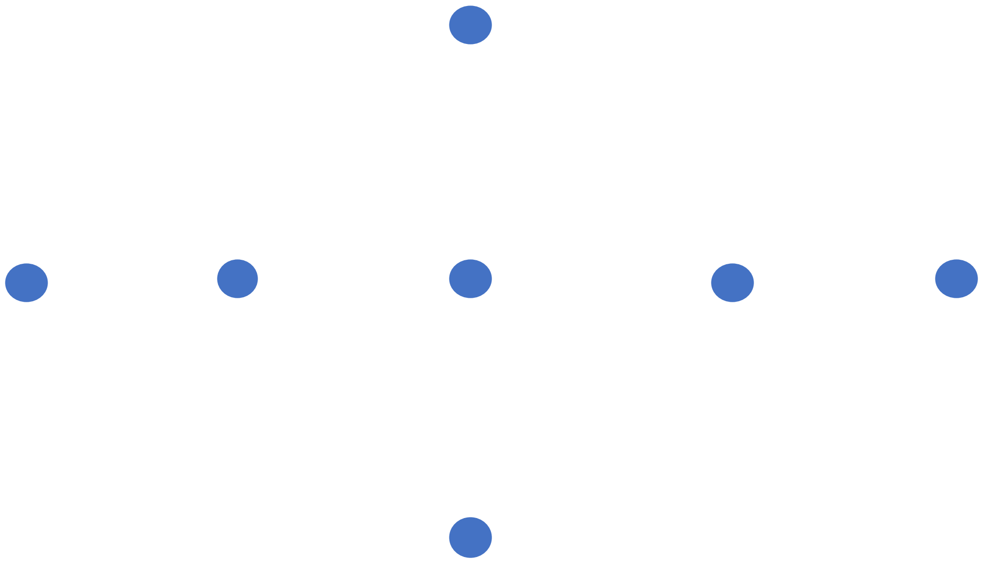

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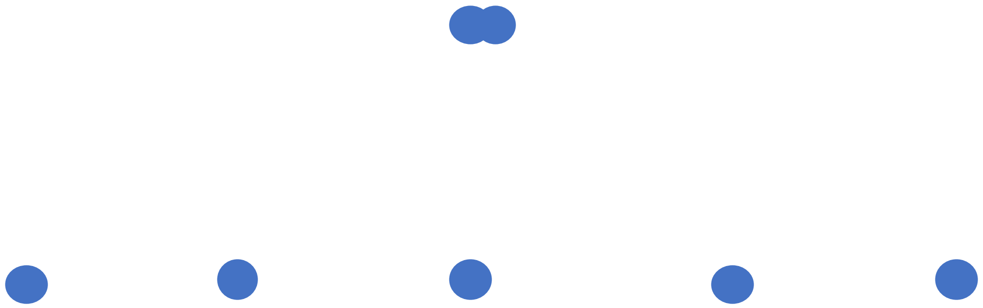

Set of ground truths with spurious local minima has positive measure

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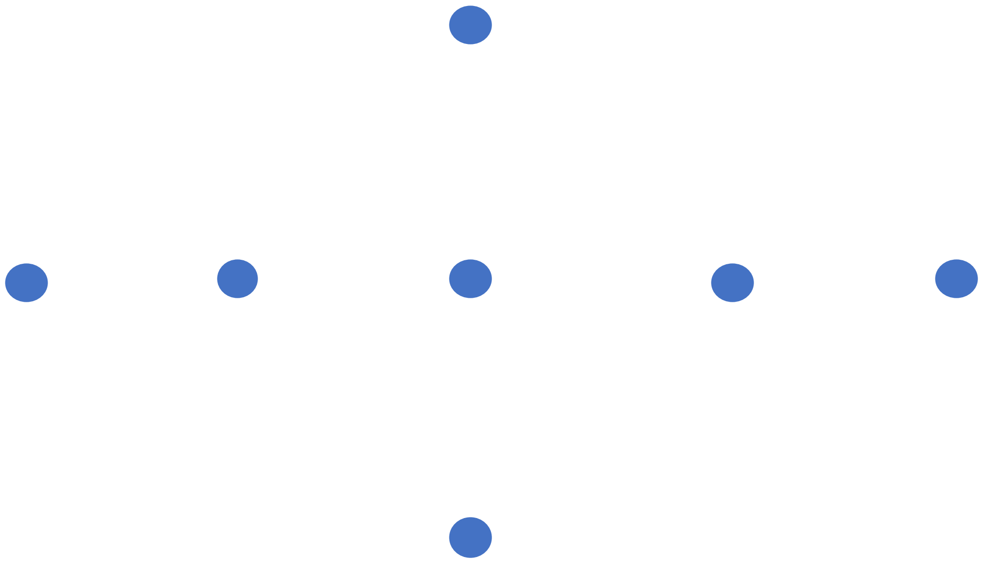

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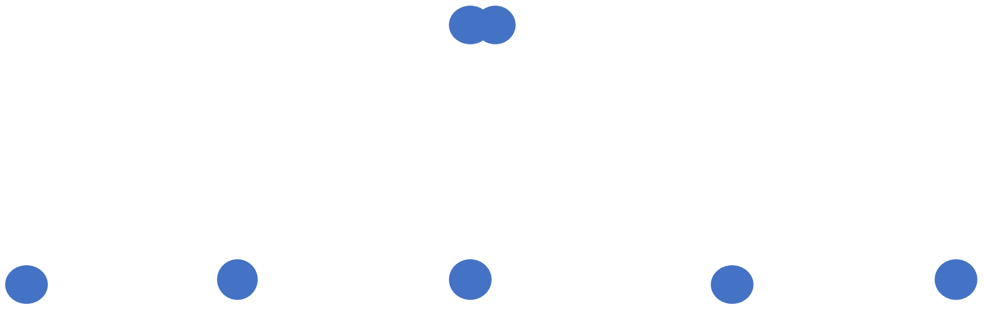

Also see:
Song, Goncalves, Jung, Lavor, Mucherino, Wolkowicz, 2024

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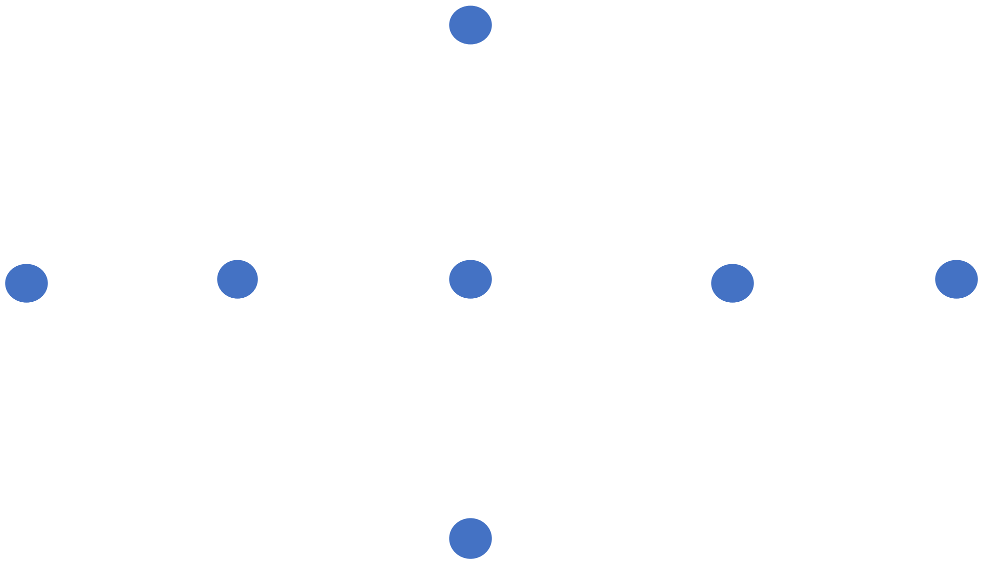

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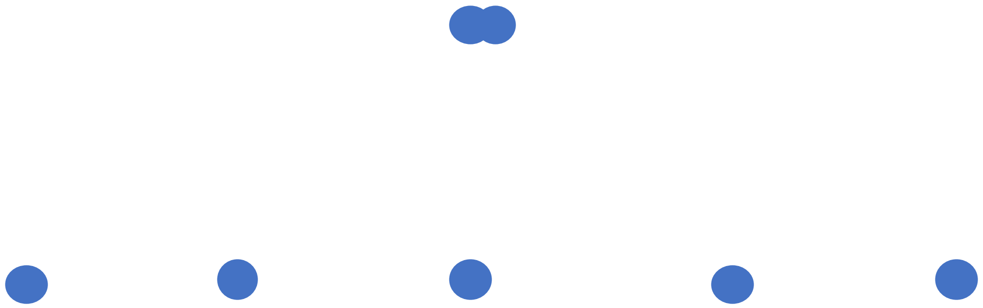

Hmm … what should we do?

Counterexample

$$\min \sum_{ij \in E} \left(\|z_i - z_j\|^2 - d_{ij}^2 \right)^2, \qquad d_{ij} = \|z_i^* - z_j^*\|$$

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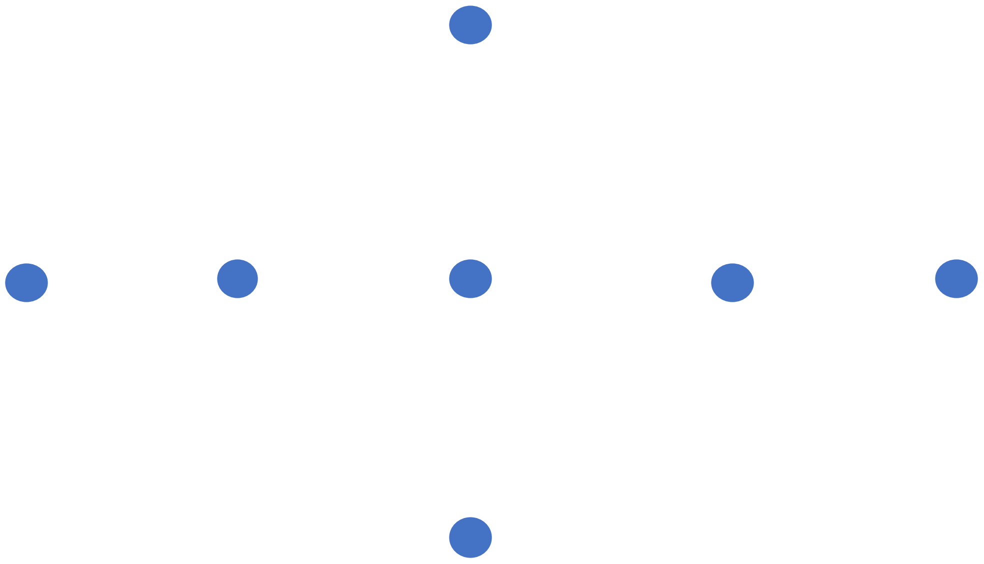
"s-stress"

s-stress can have spurious strict local minima!

Ground truth $z_1^*, z_2^*, \ldots$



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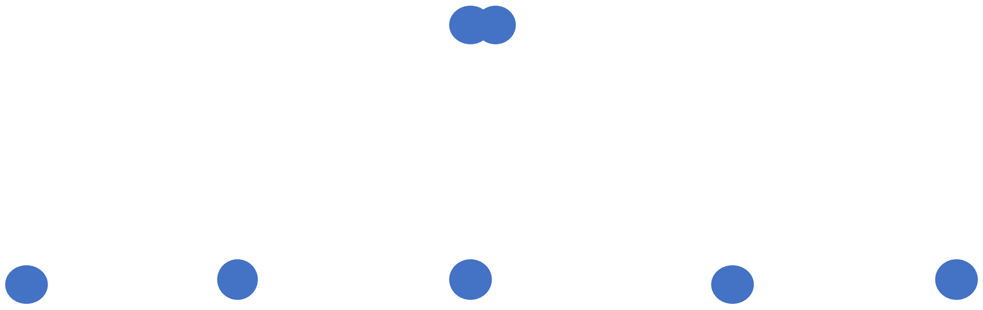

Optimize over points in $\mathbb{R}^3$

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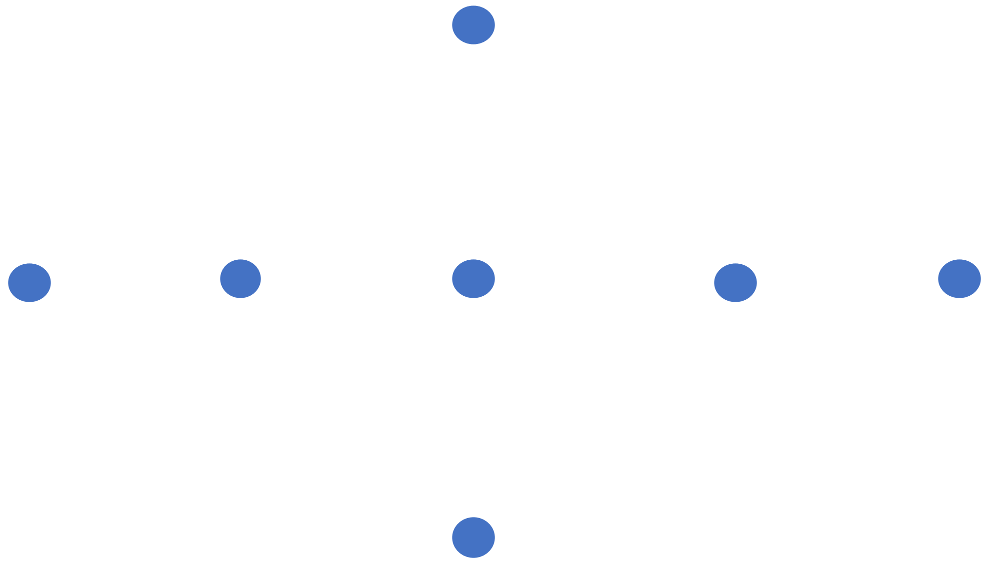

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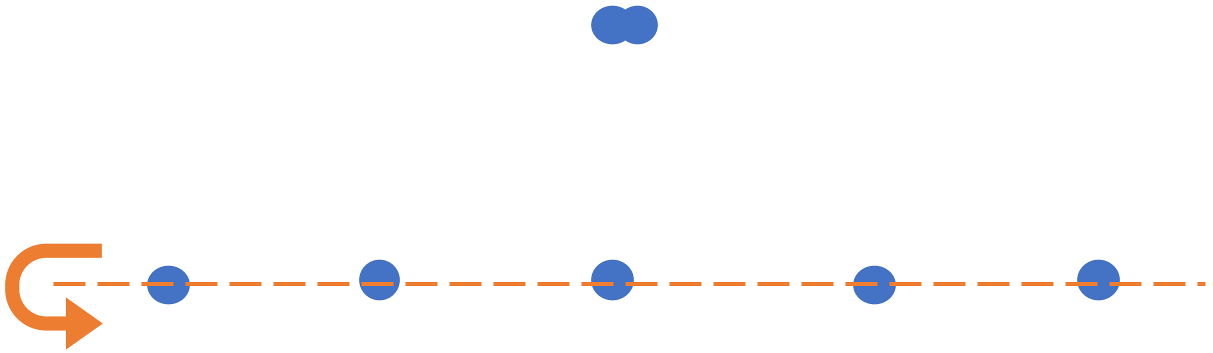

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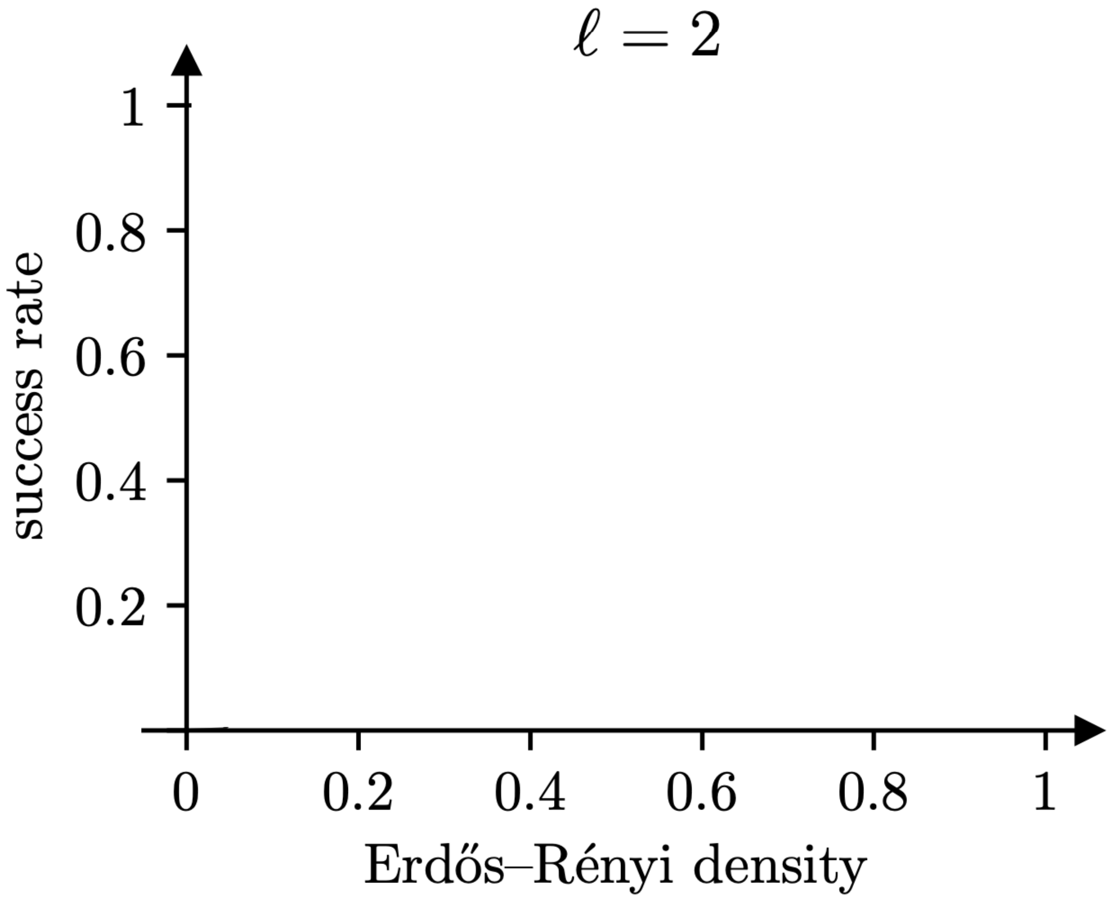
- $n = 50, \ell =$ dimension $= 2$
- Ground truth = iid Gaussian points
- Graph = ER
- Run TR from random initialization



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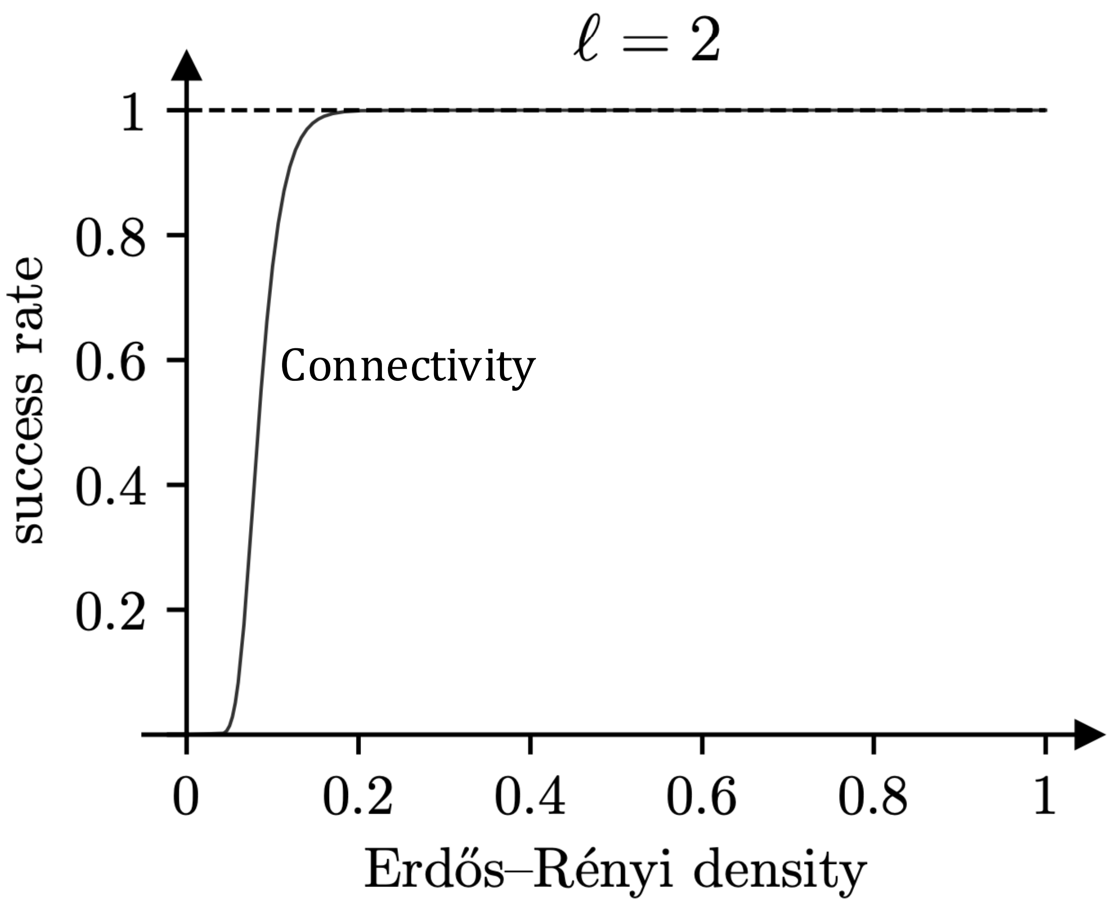

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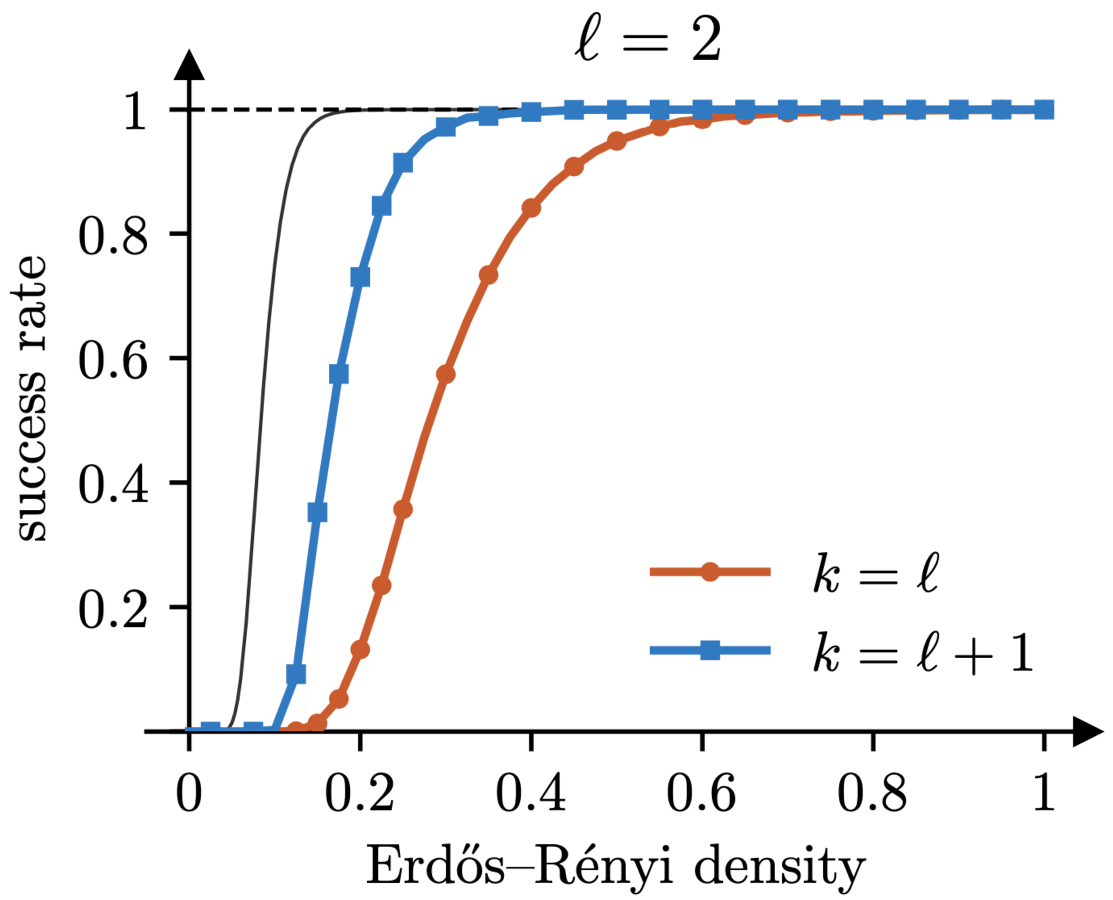

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Want k small; new problem has kn variables
If $k = n - 1$, easy to see landscape is benign (Song, Goncalves, Jung, Lavor, Mucherino, Wolkowicz, 2024)
Can we do better?

Results

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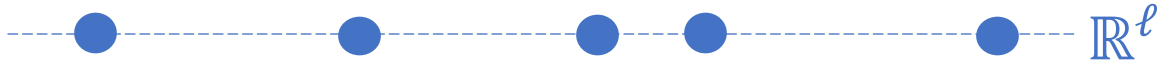

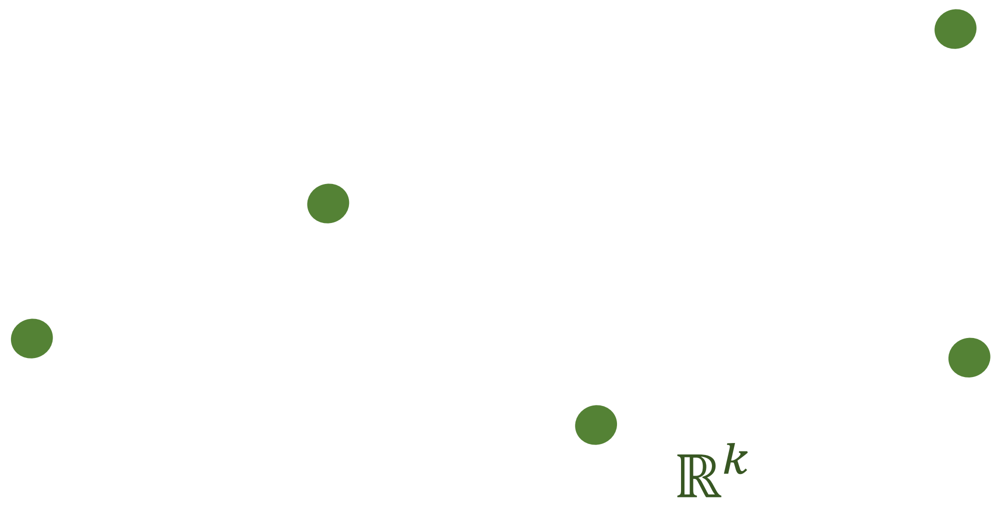

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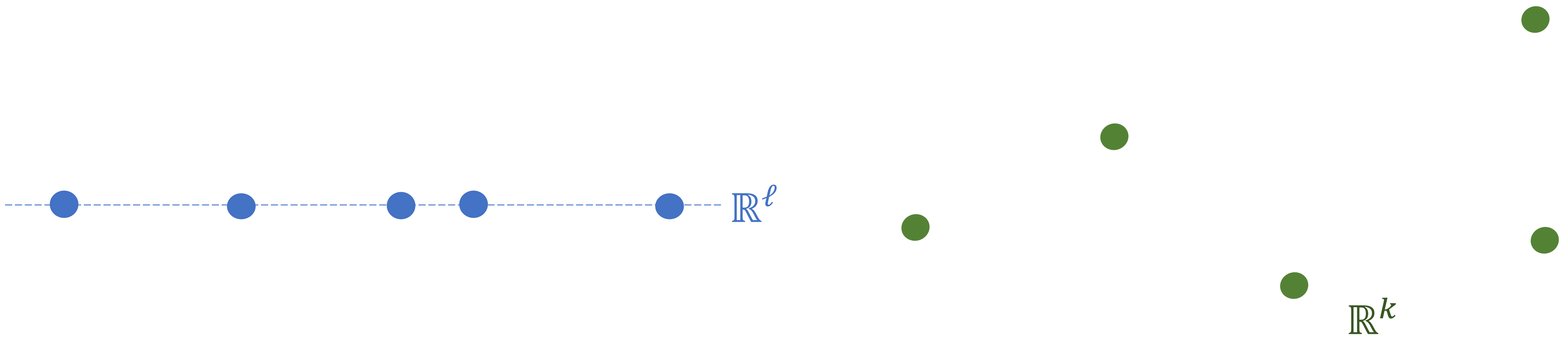

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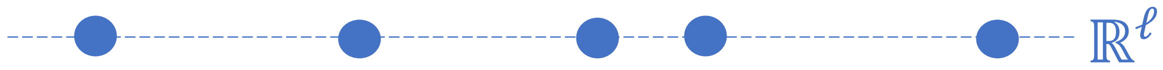

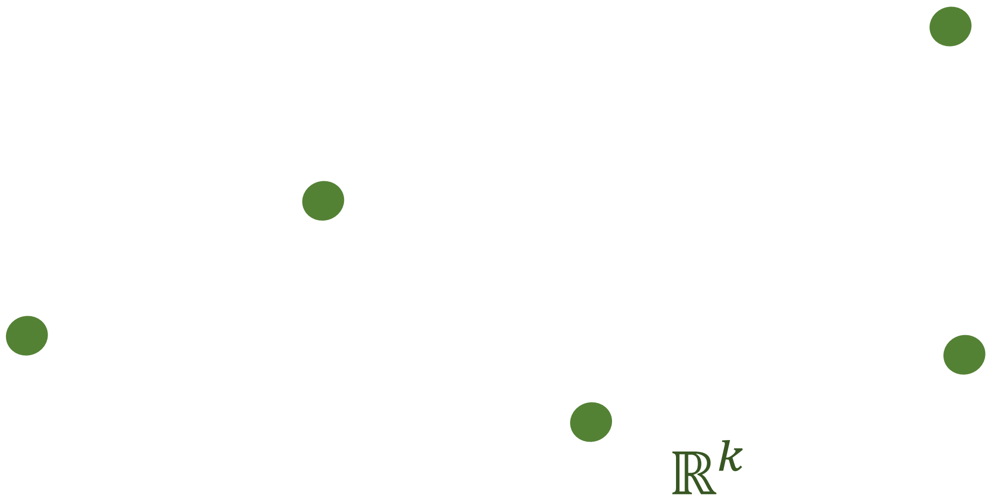

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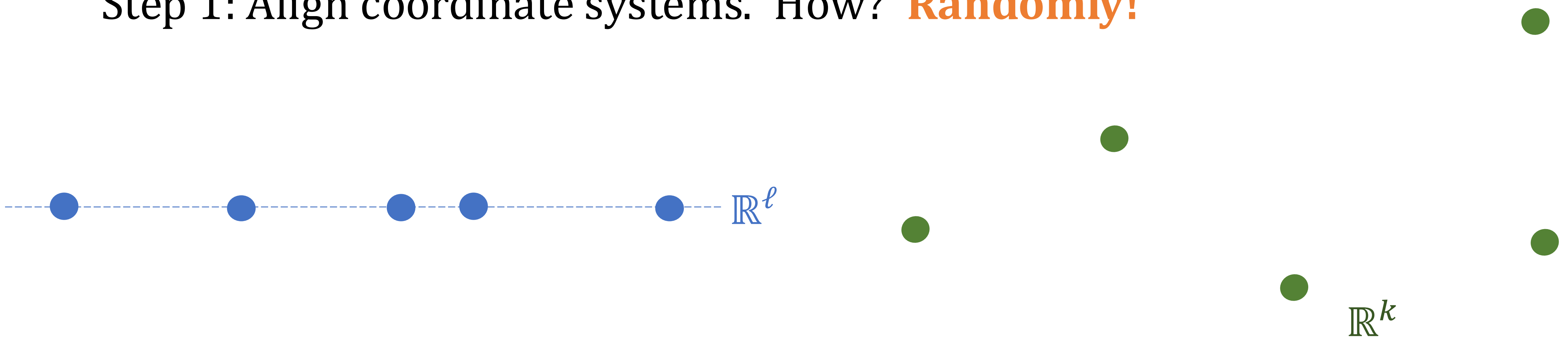

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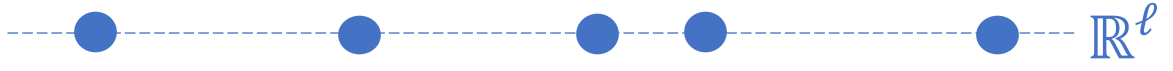

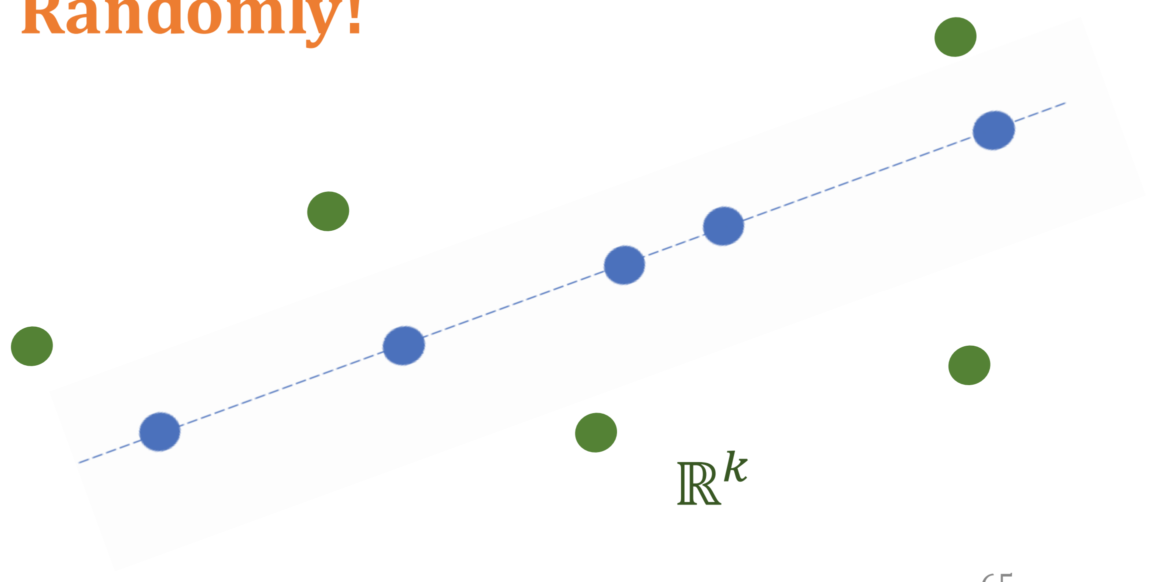

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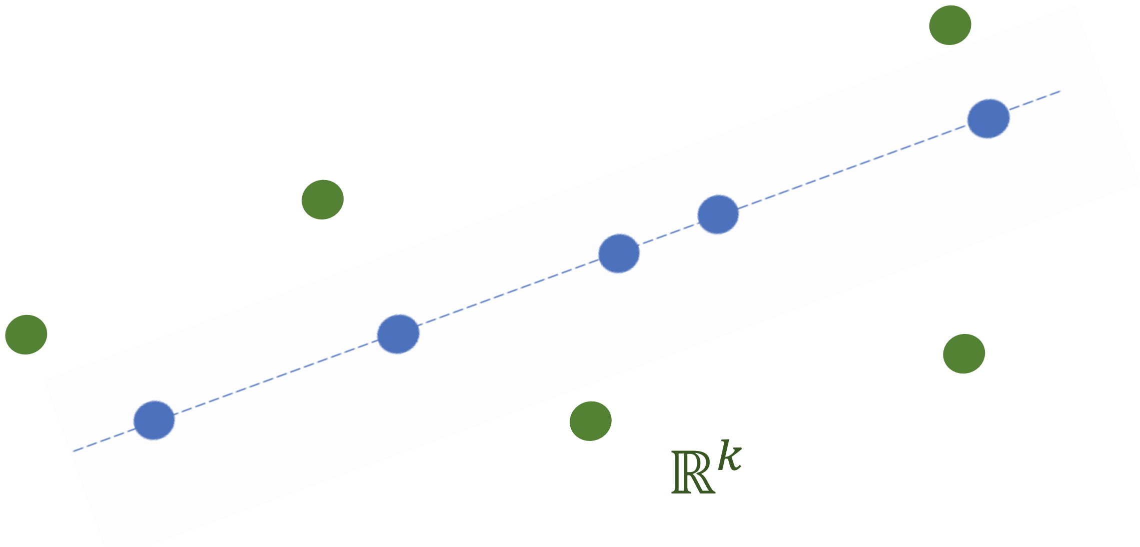

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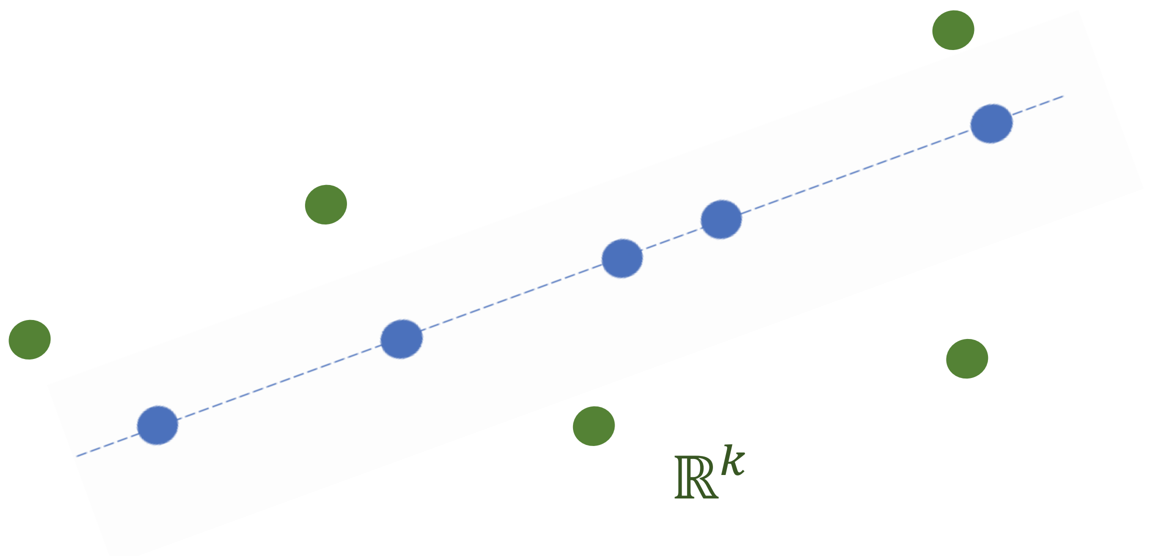

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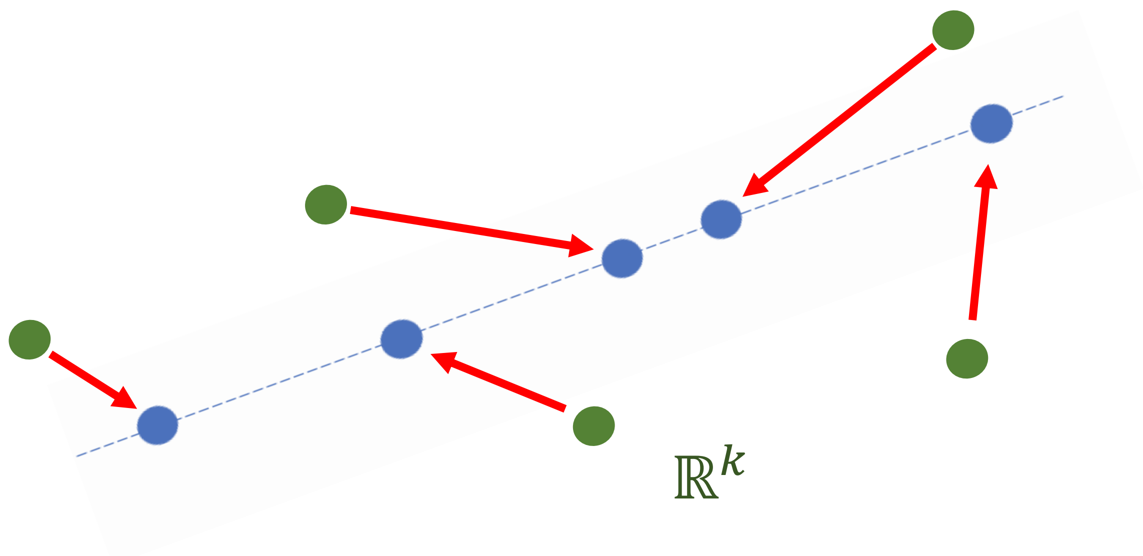

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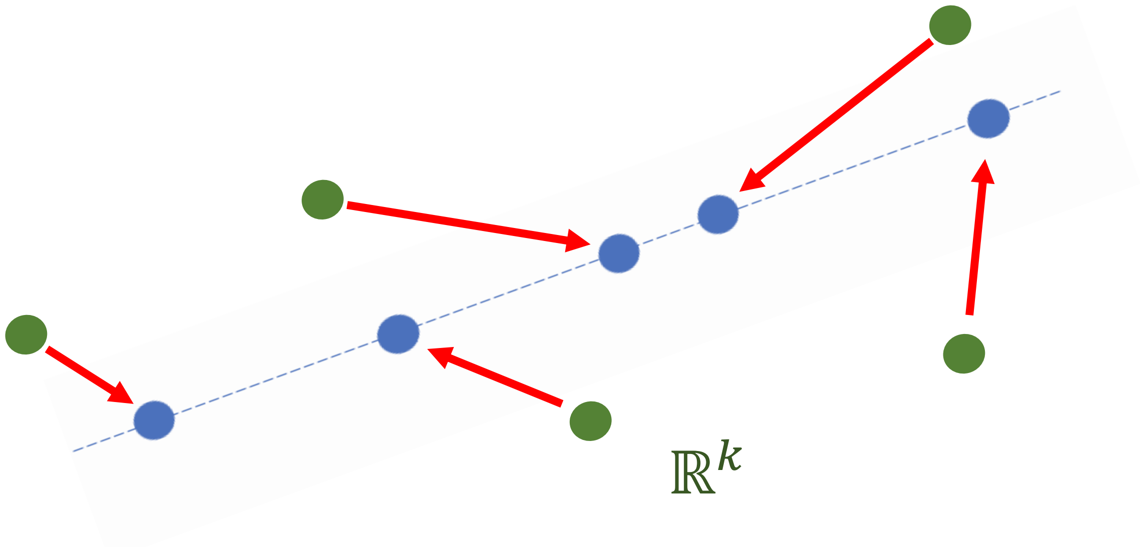

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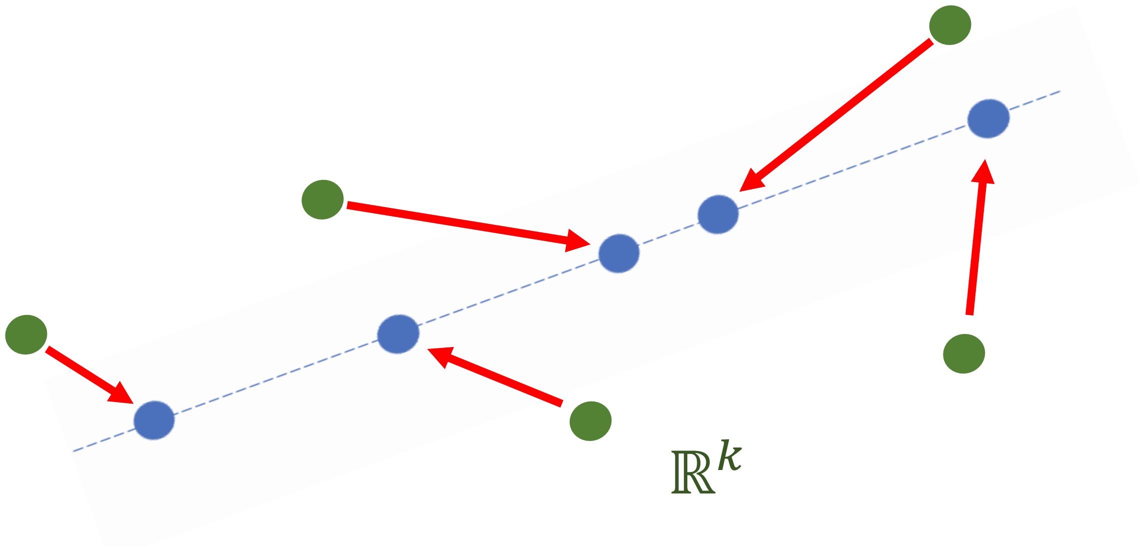

Mei, Misiakiewicz, Montanari, Oliveira '17

McRae, Boumal '23

McRae, Abdalla, Bandeira, Boumal '24

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Alternative perspective:

Low-Rank Optimization

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Gram $\to$ EDM (euclidean distance matrix)

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[Levin , Kileel, Boumal 2022; Ha, Liu, Barber 2018]

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- Conclusion: Landscape benign if $k = n$

Restricted Isometry Property?

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If RIP, then benign landscape [Bhojanapalli et al., 2016; Ge et al., 2017; Zhang et al., 2019]

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Δ does not satisfy RIP! Δ has RIP-condition-number n

Special properties of SNL map?

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Special “perturbation” of the identity

$$(\Delta^* \circ \Delta)(Y) = Y + \Psi(Y)$$

$$(\Delta^* \circ \Delta)^{-1}(Y) = Y - \Gamma(Y)$$

Special properties of SNL map?

$$\min \|\Delta(Y - Y_*)\|^2 \text{ over } Y \succcurlyeq 0 \text{ with } \mathrm{rank}(Y) \leq k$$ "s-stress"

Special "perturbation" of the identity

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New "general" theorem: If $\Gamma(Y) = \sum_{i=1}^{N} a_i a_i^\top (a_i^\top Y a_i)$ with $a_i \in \mathbb{R}^n$,

- is contractive (trace and operator norm),
- and satisfies $\langle Y, \Psi(Y)\rangle \leq c\langle Y, \Gamma(Y)\rangle \quad \forall Y$

then landscape is benign when relax to $k \approx \ell + \sqrt{c\ell}$.

Takeaways for SNL

Summary:

- s-stress can have spurious local mins (even for complete graph)
- If relax mildly ($\sqrt{n}\ \ or \log n$), s-stress landscape becomes benign

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Conceptual takeaways:

- Low-dimensional nonconvex relaxations (cheap and often work!)
    - Other applications?
- Randomized directions for proving benign landscapes
- Going beyond RIP: structured "perturbations"

Taking a step back

- Most landscape results proved on case-by-case basis

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$\min y^\top a$ over y in simplex $\xrightarrow{y = \phi(z) = z \odot z}$ $\min z^\top A z$ over z in sphere

$A = \mathrm{Diag}(a)$

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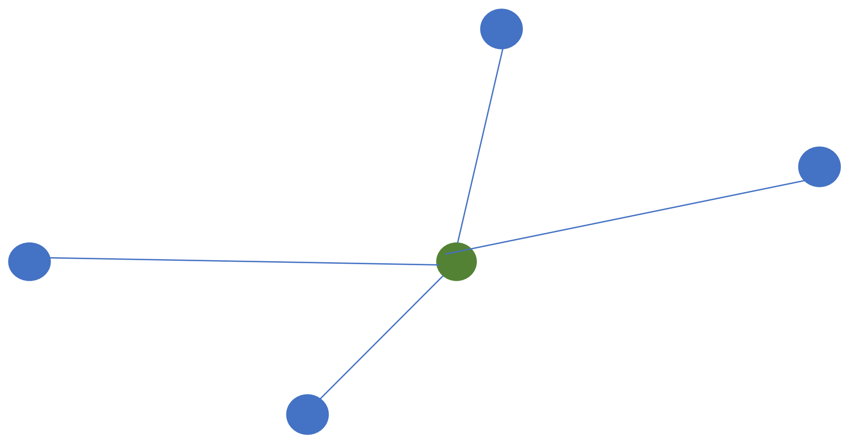
Empirically, relaxing removes critical points of index 1.

Appendix

SNL with landmarks

$$\min \sum_i \left(\|z - z_i\|^2 - d_i^2\right)^2, \qquad d_i = \|z^* - z_i^*\|$$

over $z \in \mathbb{R}^\ell$



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over $z \in \mathbb{R}^\ell$

Landscape is not benign in general.

Proposition: If relax to $k = \ell + 1$, the landscape is benign.

Hubs

Theorem [isotropic GT]: If graph is **nearly complete**, ground truth points are isotropic and iid, and relax to

$$k \approx \ell \log(n)\,,$$

then every 2-critical point is the ground truth.

The **hub** of a graph is the set of vertices which are connected to all other vertices.

$$H = \text{size of hub}$$

Theorem [isotropic GT]: If ground truth points are isotropic and iid, and relax to

$$k \approx \text{poly}(n - H)\ell \log(n)\,,$$

then every 2-critical point is the ground truth.

Counterexamples

Minima number of points to have spurious local minima?

$$n = \ell + 2 \text{ (for } \ell \geq 5)$$